Stochastic Dual Dynamic Programming
Applied to Nonlinear Hydrothermal Models

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• Nonlinear constraints reformulation
  – McCormick envelope for bilinear terms
  – Disjunctive programming

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• Case Study and Numerical Results

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Introduction

- **Linear** hydrothermal model
  - Minimize total operating cost while satisfying demand for power
  - Constant hydro production function
- **Advantages**
  - Possibility of using LP-solvers
  - Monotonically decreasing and convex water value function
  - Solutions of large-scale stochastic models using decomposition techniques
- **Disadvantage**
  - Multiplicity of solutions for reserve profiles
  - Inadequate profiles
Introduction

- A non linear hydrothermal model
  - Non constant hydro production function
  - Increases the production of the hydro plant with the head of the reservoir
- The hydro production function
  - Power (MW) = Water discharge (m$^3$/s) \cdot Head (m) \cdot Efficiency of hydro unit (head)
Introduction

- A non linear hydrothermal model
  - Non constant hydro production function
  - Increases the production of the hydro plant with the head of the reservoir
- The hydro production function
  - $\text{Power (MW)} = \text{Water discharge (m}^3/\text{s}) \cdot \text{Head (m)} \cdot \text{Efficiency of hydro unit (head)}$

$$P = q \cdot h \cdot \eta(h)$$

Simplification as an affine function

$$P = q(\alpha + \beta h)$$
Introduction

- **Bilinear relations** for modeling hydro production functions
  - Forces the use of nonlinear solvers
  - Possibility of stacking in a local minima
  - More computation time
  - Nonconvex recourse function
  - Difficulty of applying decompositions techniques
  - Difficulty of solving the stochastic problem
- **Example** of nonconvex recourse function
Introduction

Cost

Reserve level
Introduction

¿How to extend the Stochastic Dual Dynamic Programming decomposition technique to deal with this situation?
Nonlinear constraints reformulation

- Reformulate the bilinear terms using **McCormick reformulation**

\[
\begin{align*}
  z &= xy \\
  z &\geq x\bar{y} + \bar{x}y - \bar{x}y \\
  z &\geq xy + \underline{x}y - xy \\
  z &\leq x\bar{y} + xy - xy \\
  z &\leq xy + \bar{x}y - \bar{x}y
\end{align*}
\]

- Enables the use of **LP solvers**
Nonlinear constraints reformulation

Hydro production function

Water discharge

$\times 10^8$

$0$

$0.5$

$1$

$1.5$

$2$

$2.5$

$3$

$3.0$

250

50

100

200

300

500

800

1000

Stochastic Dual Dynamic Programming applied to Nonlinear Models
Nonlinear constraints reformulation
Nonlinear constraints reformulation
Nonlinear constraints reformulation

![Graph showing water discharge and hydro production function]
Nonlinear constraints reformulation
Nonlinear constraints reformulation

- A single McCormick envelope can be insufficient
- Construction of a grid for the variables of the bilinear relation
- Construction of the McCormick envelope for each rectangle of the grid
- **Disjunctive programming** forces the model to select just one tetrahedron out of the total
- Mathematical formulation using **binary variables** and a **big-M approach**
Nonlinear constraints reformulation

\[
\begin{align*}
    z &\geq x y^m + x^n y - u^{n,m} x^n y^m - (1 - u^{n,m}) K_{1}^{n,m} \\
    z &\geq x y^m + x^n y - u^{n,m} x^n y^m - (1 - u^{n,m}) K_{2}^{n,m} \\
    z &\leq x y^m + x^n y - u^{n,m} x^n y^m - (1 - u^{n,m}) K_{3}^{n,m} \\
    z &\leq x y^m + x^n y - u^{n,m} x^n y^m - (1 - u^{n,m}) K_{4}^{n,m}
\end{align*}
\]

- We determine the most accurate big-M values that enter in above constraints
Nonlinear constraints reformulation
Nonlinear constraints reformulation
Stochastic Dual Dynamic Programming

- **Multiperiod**
- **Stochasticity given by means of a recombining tree**

\[
\begin{align*}
\min z &= c^1 x^1 + E_{\xi^2} \left[ \min c^2 x^2 + E_{\xi^3} \left[ \min c^3 x^3 + \cdots \right] \right] \\
A x^t &\leq b^t \quad t : 1, \ldots, T \\
B^t (x^t) &= d^t \quad t : 1, \ldots, T \\
T x^t + W^{t+1} x^{t+1} &= h^{t+1}(\xi^{t+1}) \quad t : 1, \ldots, T - 1
\end{align*}
\]
Stochastic Dual Dynamic Programming

• **Traditional decomposition** in master problem and subproblem

**Master Problem**

\[
\begin{align*}
\min \ z &= c^1 x^1 + E_{\xi^2} \left[ Q^1 (x^1, \xi^2) \right] \\
\text{s.t.} : \ A x^1 &\leq b^1, \ B^1 (x^1) = d^1
\end{align*}
\]

**Subproblem**

\[
Q^{t-1} (x^{t-1}, \xi^t) = \min c^t x^t + E_{\xi^t} \left[ Q^t (x^t, \xi^{t+1}) \right] \\
Ax^t &\leq b^t \\
B^t (x^t) &= d^t \\
Wx^t &\leq h(\xi^t) - T x^{t-1}
\]

**Primal Proposals**

**Outer approximations**
Stochastic Dual Dynamic Programming

\[
Q^{t-1}(x^{t-1}, \xi^t) = \min c^t x^t + E_{\xi^t} [Q^t(x^t, \xi^{t+1})]
\]
\[
A x^t \leq b^t
\]
\[
B^t(x^t) = d^t
\]
\[
Wx^t \leq h(\xi^t) - T x^{t-1}
\]

Bilinear relations
Nonlinear subproblem
Non convex recourse function

\[
Q^{t-1}(x^{t-1}, \xi^t) = \min c^t x^t + E_{\xi^t} [Q^t(x^t, \xi^{t+1})]
\]
\[
A x^t \leq b^t
\]
\[
M x^t = d^t
\]
\[
Wx^t \leq h(\xi^t) - T x^{t-1}
\]

MCormick refomulation
Linear subproblem
Convex recourse function
Slack approximation

\[
Q^{t-1}(x^{t-1}, \xi^t) = \min c^t x^t + E_{\xi^t} [Q^t(x^t, \xi^{t+1})]
\]
\[
A x^t \leq b^t
\]
\[
M x^t + N u^t = d^t
\]
\[
Wx^t \leq h(\xi^t) - T x^{t-1}
\]

MCormick surface
MIP subproblem
Non convex recourse function
Tight approximation
Stochastic Dual Dynamic Programming

- Convexification of the recourse function using Lagrangean Relaxation

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w(x^{t-1}, \xi^t, \lambda^t) = \min c^t x^t + E_{\xi^t} [Q^t (x^t, \xi^{t+1})] + \lambda^t (T x^{t-1} + W x^t - h(\xi^t))$</td>
<td>Nonlinear Subproblem</td>
</tr>
<tr>
<td>$A x^t \leq b^t$</td>
<td>Local Minima</td>
</tr>
<tr>
<td>$B^t (x^t) = d^t$</td>
<td></td>
</tr>
<tr>
<td>$w(x^{t-1}, \xi^t, \lambda^t) = \min c^t x^t + E_{\xi^t} [Q^t (x^t, \xi^{t+1})] + \lambda^t (T x^{t-1} + W x^t - h(\xi^t))$</td>
<td>MIP Subproblem</td>
</tr>
<tr>
<td>$A x^t \leq b^t$</td>
<td>Use the Best Bound</td>
</tr>
<tr>
<td>$M^t x^t + N u^t = d^t$</td>
<td></td>
</tr>
</tbody>
</table>
Stochastic Dual Dynamic Programming

- We adopt the reformulation given by the McCormick Surface for the convexification routine
- We avoid the large number of Lagrangean Relaxation iterations for the optimization of the dual function
- We chose a proper multiplier and perform just one evaluation of the Lagrangean subproblem
  - Heuristic 1. Solution of the McCormick envelope subproblem and obtain the dual variable of the coupling constraints. Set the optimal multiplier
    \[ \lambda^t = -\pi^t \]
  - Heuristic 2. Combine the coefficients of previously computed Benders cuts to create the proper multiplier
Stochastic Dual Dynamic Programming

- An example for a two stage situation
Stochastic Dual Dynamic Programming

- An example for a two stage situation
Stochastic Dual Dynamic Programming

- An example for a two stage situation
Stochastic Dual Dynamic Programming

• An example for a two stage situation
Stochastic Dual Dynamic Programming

- Description of the multistage situation
- Forward pass
  - Sample a scenario (path from the root through the tree)
  - Solve each node of the scenario (MIP subproblem)
  - Store the primal solution and the coefficients of the active Benders cuts
Stochastic Dual Dynamic Programming

- Description of the multistage situation
- Backward pass
  - Solve each node of each period
  - Create the proposed multiplier
  - Evaluate the Lagrangean subproblem (MIP)
  - Store the objective function and create a new Benders cut
Stochastic Dual Dynamic Programming

- Stopping criteria
  - Lower Bound: solution of the root node
  - Upper Bound: random variable. Estimation after n scenarios together with a confidence interval

\[
\bar{z} = \frac{1}{N} \sum_{t=1}^{T} c_t(x_t^{\xi})^n \quad \sigma_n = \sqrt{\frac{1}{N} \sum_{n=1}^{N} \left( \sum_{t=1}^{T} c_t(x_t^{\xi})^n - \bar{z} \right)^2}
\]

\[
I = \left( \bar{z} - \frac{1.96}{\sqrt{N}} \sigma_n, \bar{z} + \frac{1.96}{\sqrt{N}} \sigma_n \right)
\]

- Stopping rule:
  - Lower bound within the confidence interval
  - Confidence interval with a given tolerance
Case Study

- **Real size** hydrothermal coordination problem
- **One year** planning horizon
- **Weekly** period representation
- 84 thermal units
- 24 hydro plants
- 3 basins and multiple **cascade reservoirs**
- Recombining scenario tree created with **clustering techniques**
- Approximation of the bilinear relation with the McCormick surface with different pieces for the hydro production variable and the water discharge variable
Case Study

- Practical implementation of the decomposition method
- Phase 1
  - Forward and backward solution of the linear relaxation of the node subproblems
- Phase 2
  - Forward solution of the linear relaxation of the node subproblems.
  - Backward solution of Lagrangean subproblem evaluations. Multiplier proposed combining the coefficients of the active cuts in the forward pass (heuristic 2)
- Phase 3
  - Forward solution of the MIP subproblems.
  - Backward solution of the Lagrangean subproblem evaluations using heuristic 1
Case Study

- Convergence evolution
## Case Study

<table>
<thead>
<tr>
<th>Branching</th>
<th>Sc</th>
<th>Lower</th>
<th>Upper</th>
<th>Interval</th>
<th>Tolerance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Every 4 weeks</td>
<td>$2^{12}$</td>
<td>7821.0</td>
<td>7817.7</td>
<td>[7803.5, 7831.9]</td>
<td>0.0036</td>
</tr>
<tr>
<td>Every 2 weeks</td>
<td>$2^{25}$</td>
<td>7828.0</td>
<td>7839.3</td>
<td>[7826.4, 7852.2]</td>
<td>0.0033</td>
</tr>
<tr>
<td>Every 1 week</td>
<td>$2^{51}$</td>
<td>7839.0</td>
<td>7850.3</td>
<td>[7831.6, 7868.9]</td>
<td>0.0047</td>
</tr>
<tr>
<td>Every 4 weeks</td>
<td>$3^{12}$</td>
<td>7828.6</td>
<td>7830.8</td>
<td>[7791.7, 7869.9]</td>
<td>0.0099</td>
</tr>
</tbody>
</table>
Case Study: Evolution of the reserve profiles
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Case Study: Evolution of the reserve profiles
Case Study: Evolution of the reserve profiles
Conclusions

• Extension of the Stochastic Dual Dynamic Programming algorithm for nonlinear subproblems reformulated as MIP subproblems
• Remarkable results for the hydrothermal coordination problem. Acceptable reserve profiles
• Future developments
  – Sensitivity analysis for the uncertainty representation and the grid precision for the McCormick Surface
  – Adjust the stopping rule criteria with the theory developments in literature
  – Incorporate variance reduction techniques to reduce the computation time
  – Explore the possibility of performing the algorithm by solving small recombining subtrees during the iterations
  – Risk constraints for risk control of spillages
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