Mathematical programming approach to underground timetabling for maximizing the use of regenerative braking power

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Keywords: train timetabling problem, train scheduling, energy saving.

1. Introduction

Underground transportation is crucial in big modern cities as a way to achieve a clean, rapid and massive transport. While in peak hours the first objective is to move as many people as possible increasing the train frequency, in off-peak hours other considerations can be taken into account. Energy consumption should be an important issue in the design of train timetables in off-peak hours. Energy saving can be obtained by using regenerative brakes.

The model presented here is a particular case of train timetabling problem. It is stated as a mixed integer optimization problem. The objective function is to maximize the overlapping time among trains that arrive and depart from the same station or from different stations connected to the same electrical substation. Under that condition the energy produced by a train in its slow-down process can be consumed by another close train in its speed-up process. The constraints include maximum bounds on changes in the current timetable in service with respect to the new synchronised schedule, keeping the total travel time of each train, and the computation of the coincidence time among trains. The detection of the overlapping condition requires binary variables and, therefore, integrality conditions, which make the problem difficult to solve. A mathematical programming and a genetic algorithm approaches have been explored. In Bussieck (1997) and Nielsen (2006) can be found mixed integer programming formulations solved directly in the general context of railroad operation timetabling. For other approaches based on metaheuristic techniques see Godwin (2006), where they resort to genetic algorithms.

The train timetabling model has been tested with real cases corresponding to lines of Metro de Madrid and the preliminary results show that time coincidence can be increased dramatically and, therefore, energy saving by train synchronization.

The paper is organized as follows. In section 2 the rational of the model is presented. Then, the mathematical formulation of the optimization problem is stated in section 3. In section 4, a case study is developed corresponding to line 3 of Metro de Madrid. Finally, some conclusions are summarized and some future extensions are suggested in the last section.

2. Model description

The model presented here is a particular case of train timetabling problem. Its purpose is maximizing the overlapping time between speed-up and slow-down actions of all the trains.
circulating at any time and located in the same electrical section. As said in the introduction, the model is applied only to trains running during off-peak hours (after 23 h) because night train schedules can be observed almost strictly due to lack of incidences. This model is a useful tool to define the off-peak timetables where energy saving can be an important goal in train scheduling. An experience in the Rome (Italy) underground has reported an energy saving of 15% due to the use of regenerative braking without a synchronization objective, see Adinolfi (1998). The synchronization problem has also been addressed in the context of real-time operation in Albrecht (2002).

The initial schedule is taken as given and the model maximizes the coincidence time while satisfying several operating constraints in order to obtain an implementable timetable.

The problem is stated as a mixed integer programming (MIP) problem. Two approaches have been followed to solve the problem: firstly, a mathematical programming approach and secondly, a hybrid approach where mutations were applied to solutions obtained with the first approach.

Several uses of interest can be devised when solving the optimization problem: i) evaluation of the overlapping time for the initial timetable, ii) maximizing overlapping time keeping an published commercial timetable that exclusively determines departure times.

3. Mathematical formulation

3.1. Indices

\( i \) train. Trains are supposed to do just a round trip from the beginning to the ending station and then back.

\( j \) platform (for example, northbound and southbound) of an underground station. \( j = 1, \ldots, J \), being \( 1, J/2, J/2 + 1, J \) terminal platforms and \( 1, J/2 + 1 \) head platforms.

3.2. Parameters

The following data are supposed to be known in advance and correspond to the initial timetable, to the duration of the slow-down and speed-up processes, and to some adjustment parameters that avoid dramatic changes in the final schedule. Lower case letters are used to define the parameters.

\( d_{ij} \) initial departure times of train \( i \) from platform \( j \) [s]

\( sd, su \) slow-down and speed-up times of any train at any platform\(^1\) [s]

\( s_{\text{max}, j}, s_{\text{min}, j} \) maximum and minimum stopping time at platform \( j \) [s]

\( t_{\text{max}, j}, t_{\text{min}, j} \) maximum and minimum travelling time from platform \( j-1 \) to \( j \) [s]

\(^1\) They can easily be particularized for each platform \( j \) and even train type to take into consideration their specific characteristics.
\[ \Delta d_j, \nabla d_j \] maximum and minimum change in departing time from platform \( j \) [s]

\[ \Delta tt \] maximum increment in total trip time for any train [s]

\[ p_{ji} \] penalty factor introduced to consider approximately the loss in the electricity transferred between trains at different platforms \( j \) and \( j' \) although both belong to the same electrical section [p.u.]. If two platforms belong to different electrical sections \( p_{ji} = 0 \).

3.3. Variables

The variables of the optimization problem are written in capital and Greek letters and correspond to the following ones:

- \( A_j, D_j \) arrival and departure times of train \( i \) at platform \( j \) [s]
- \( \delta_{ji} \) binary variable that indicates whether there is or not (1/0, respectively) coincidence between the slow-down of train \( i \) at platform \( j \) and the speed-up of train \( i' \) at platform \( j' \)
- \( T_{ji} \) overlapping time between the slow-down and speed-up processes of train \( i \) at platform \( j \) and train \( i' \) at platform \( j' \), respectively [s]
- \( B_j, C_j \) change in arrival and departure times of train \( i \) at platform \( j \) with respect to the initial timetable [s]

3.4. Constraints

The following constraints take into account the operating conditions of the trains.

- Stopping time for each train \( i \) at platform \( j \) has to be bounded by the corresponding bounds

  \[ s_{\min,j} \leq D_j - A_j \leq s_{\max,j} \quad \forall ij \]  
  (1)

  The stopping time of any train at the terminal station is considered to take a constant time.

- Travelling time for each train \( i \) at platform \( j \) has to be bounded by the corresponding bounds

  \[ t_{\min,j} \leq A_j - D_{j-1} \leq t_{\max,j} \quad \forall ij \]  
  (2)

  The platform change of the train at the terminal station is considered to take a constant time. This constraint is not formulated for any head station of the line.

- Change in the departure time for each train \( i \) at platform \( j \) with respect to the initial schedule has to be bounded by the corresponding bounds

  \[ \nabla d_j \leq D_j - d_j \leq \Delta d_j \quad \forall ij \]  
  (3)
- Change in the total trip time for each train $i$ and in each way with respect to the initial schedule has to be bounded by the corresponding bounds

$$
(D_{ij}^2 - D_{ij}^1) - (d_{ij}^2 - d_{ij}^1) \leq \Delta t \quad \forall i
$$
$$
(D_{ij} - D_{ij+1}^2) - (d_{ij} - d_{ij+1}^2) \leq \Delta t \quad \forall i
$$
$$
- \Delta t \leq (D_{ij}^2 - D_{ij}^1) - (d_{ij}^2 - d_{ij}^1) \quad \forall i
$$
$$
- \Delta t \leq (D_{ij} - D_{ij+1}^2) - (d_{ij} - d_{ij+1}^2) \quad \forall i
$$

One way is considered from the departure platform at the head station to same side platform at the terminal station, $J/2$, and the other way is considered from the other side platform at the terminal station $J-1$ to the opposite platform at the head station $J$. The departure time at the second platform of the terminal station represents when the train is available to begin a new trip.

- Computation of overlapping time

Before writing the constraints that allow the computation of the overlapping time, the different possibilities of train coincidence are described. Let us define $A_y = A_y - s_d$ the beginning of the slow-down process before arriving to a platform and $D_{ij}^* = D_{ij} + s_u$ the end of the speed-up process after departure of a platform. The six combinations and their overlapping time are presented in the following table, where departure times are in black colour (square brackets) and arrival times are in blue (parenthesis).

<table>
<thead>
<tr>
<th>Case</th>
<th>Sequence</th>
<th>Overlapping time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$D_{ij}$, $A_y$, $A_y$, $D_{ij}$*</td>
<td>$A_y - A_y = sd$</td>
</tr>
<tr>
<td>2</td>
<td>$A_y$, $D_{ij}$, $D_{ij}$*, $A_y$</td>
<td>$[1]$</td>
</tr>
<tr>
<td>3</td>
<td>$D_{ij}$, $A_y$, $D_{ij}$*, $A_y$</td>
<td>$[1]$</td>
</tr>
<tr>
<td>4</td>
<td>$A_y$, $D_{ij}$, $A_y$, $D_{ij}$*</td>
<td>$[1]$</td>
</tr>
<tr>
<td>5</td>
<td>$D_{ij}$, $D_{ij}$*, $A_y$, $A_y$</td>
<td>$[0]$</td>
</tr>
<tr>
<td>6</td>
<td>$A_y$, $A_y$, $D_{ij}$<em>, $D_{ij}$</em></td>
<td>$[0]$</td>
</tr>
</tbody>
</table>

For example, there is no coincidence if a train begins the slow-down process after the speed-up process of another train finishes ($A_y \geq D_{ij}$*, case 5) or if a train departs after the arrival of another train ($D_{ij} \geq A_y$, case 6). These cases can be modelled as a logical implication

$$
A_y \geq D_{ij'}^* \text{ or } D_{ij} \geq A_y \Rightarrow \delta_{yij} = 0
$$

being $\delta_{yij}$ the binary variable that indicates the coincidence condition. $\delta_{yij} = 1$ means coincidence.

This implication can be modelled by the linear constraints:
\[ A_{ij} - D_{ij}' \leq M (1 - \delta_{ij}') \quad \forall iij'j' \]
\[ D_{ij} - A_y \leq M (1 - \delta_{ij}') \quad \forall iij'j' \] (6)

being \( M = \max \left( \left| d_{ij} - d_y + s_{\max,j} \right| \right) + su + sd \) an upper bound of the constraint.

In the other cases (1 to 4), the overlapping time can be calculated as

\[
\begin{align*}
T_{ij}' &\leq M' \delta_{ij}' \quad \forall iij'j' \\
T_{ij}' &\leq D_{ij}' - D_{ij} + M (1 - \delta_{ij}') \quad \forall iij'j' \\
T_{ij}' &\leq D_{ij}' - A_y + M (1 - \delta_{ij}') \quad \forall iij'j' \\
T_{ij}' &\leq A_y - D_{ij} + M (1 - \delta_{ij}') \quad \forall iij'j' \\
T_{ij}' &\leq A_y - A_y + M (1 - \delta_{ij}') \quad \forall iij'j' 
\end{align*}
\] (8)

being the maximum overlapping time the slow-down or speed-up time of any train, \( M' = \min(su, sd) \). These constraints can be reformulated as

\[
\begin{align*}
T_{ij}' &\leq M' \delta_{ij}' \quad \forall iij'j' \\
T_{ij}' &\leq su \delta_{ij}' \quad \forall iij'j' \\
T_{ij}' &\leq D_{ij}' - A_y + M (1 - \delta_{ij}') \quad \forall iij'j' \\
T_{ij}' &\leq A_y - D_{ij} + M (1 - \delta_{ij}') \quad \forall iij'j' \\
T_{ij}' &\leq sd \delta_{ij}' \quad \forall iij'j' 
\end{align*}
\] (9)

It can be observed that the third and fourth equations of the set (9) and the condition of non negative overlapping time \( T_{ij}' \geq 0 \) turn superfluous equations (6). In the same way, the second and fifth equations of set (9) can be substituted by the first equation of this set.

\[
\begin{align*}
T_{ij}' &\leq M' \delta_{ij}' \quad \forall iij'j' \\
T_{ij}' &\leq D_{ij}' - A_y + M (1 - \delta_{ij}') \quad \forall iij'j' \\
T_{ij}' &\leq A_y - D_{ij} + M (1 - \delta_{ij}') \quad \forall iij'j' 
\end{align*}
\] (10)

Finally, a set of equations are added to avoid changes in the timetable that do not improve the overlapping time

\[ -B_y \leq D_y - d_y \leq B_y \quad \forall ij \] (11)

where variable \( B_y \) corresponds to changes in departure time with respect to the initial timetable. The variable is introduced in the objective function with a very small penalty \( \epsilon \).

3.5. Objective function

The objective function maximizes the total overlapping time multiplied by the penalty factor introduced to consider the electricity losses.
\[ \max \sum_{y,j} p_{y,j} T_{y,j} - \varepsilon \sum_{y} B_{y} \quad (12) \]

### 3.6. Mathematical problem

The MIP optimization problem that maximizes the total overlapping time between the slow-down and speed-up processes of different trains can be stated as

\[
\begin{align*}
\max & \quad \sum_{y,j} p_{y,j} T_{y,j} - \varepsilon \sum_{y} B_{y} \\
\text{s.t.} & \quad s_{\min,j} \leq D_{y} - A_{y} \leq s_{\max,j} \quad \forall ij \\
& \quad t_{\min,j} \leq A_{y} - D_{y-1} \leq t_{\max,j} \quad \forall ij \\
& \quad \nabla d_{j} \leq D_{y} - d_{y} \leq \Delta d_{j} \quad \forall ij \\
& \quad \left(D_{y/j_{2}} - D_{y}\right) - \left(d_{y/j_{2}} - d_{y}\right) \leq \Delta t \quad \forall i \\
& \quad \left(D_{y} - D_{y/j_{2}+1}\right) - \left(d_{y} - d_{y/j_{2}+1}\right) \leq \Delta t \quad \forall i \\
& \quad T_{y/j'} \leq M^{*}\delta_{y/j'} \quad \forall ij'j' \\
& \quad T_{y/j'} \leq D_{y/j'} - A_{y} + M(1-\delta_{y/j'}) \quad \forall ij'j' \\
& \quad T_{y/j'} \leq A_{y} - D_{y/j'} + M(1-\delta_{y/j'}) \quad \forall ij'j' \\
& \quad -B_{y} \leq D_{y} - d_{y} \leq B_{y} \quad \forall ij \\
& \quad A_{y}, D_{y/j'}, T_{y/j'}, B_{y} \geq 0, \delta_{y/j'} \in \{0,1\}
\end{align*}
\]

\[ \quad (13) \]

The size of the problem is parameterized in this table and estimated for \( I = 15 \) night trains, and \( J = 36 \) platforms (corresponding to 18 stations).

<table>
<thead>
<tr>
<th></th>
<th>( I = 15 ), ( J = 36 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constraints</td>
<td>( 8IJ + 2I^2J^2 )</td>
</tr>
<tr>
<td>Continuous variables</td>
<td>( 3IJ + I^2J^2 )</td>
</tr>
<tr>
<td>Binary variables</td>
<td>( I^2J^2 )</td>
</tr>
</tbody>
</table>

To avoid the curse of dimensionality in the size of the problem the possible combinations between different trains at platforms can be substantially reduced by dealing only with those trains that are relative close in the original timetable, \( ij'j' \in c(i,j,i',j') \), being \( c(i,j,i',j') \) the set of close trains. For example, in the case study presented in the following section the size of the problem is approximately 9370 constraints, 6420 continuous variables and 2120 binary variables very far from the previous estimation. The set of close trains is determined by the model given a time scalar specified by the user.

### 3.7. MIP Approach

The model has been written in GAMS 22.8, see Brooke (2008). In a first approach the model has been solved by CPLEX 11.1, see ILOG, under a PC at 2.40 GHz with 504 MB of RAM memory running the Microsoft Windows XP operating system. A Microsoft Excel interface has been used for input data and output results.

In the following table some results of the mathematical problem for different maximum solution times are presented.
### Table

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>8.840</td>
<td>11.073</td>
<td>0.20</td>
</tr>
<tr>
<td>3.4</td>
<td>8.840</td>
<td>11.073</td>
<td>0.20</td>
</tr>
<tr>
<td>16.0</td>
<td>9.323</td>
<td>11.071</td>
<td>0.16</td>
</tr>
<tr>
<td>29.9</td>
<td>9.659</td>
<td>11.071</td>
<td>0.13</td>
</tr>
<tr>
<td>62.1</td>
<td>10.078</td>
<td>11.067</td>
<td>0.09</td>
</tr>
<tr>
<td>63.6</td>
<td>10.078</td>
<td>11.067</td>
<td>0.09</td>
</tr>
<tr>
<td>90.0</td>
<td>10.274</td>
<td>11.067</td>
<td>0.07</td>
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<tr>
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<td>10.413</td>
<td>11.067</td>
<td>0.06</td>
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<td>101.9</td>
<td>10.581</td>
<td>11.067</td>
<td>0.04</td>
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<tr>
<td>117.0</td>
<td>10.762</td>
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<td>0.03</td>
</tr>
<tr>
<td>196.3</td>
<td>11.023</td>
<td>11.067</td>
<td>0.00</td>
</tr>
<tr>
<td>305.3</td>
<td>11.063</td>
<td>11.067</td>
<td>0.00</td>
</tr>
<tr>
<td>500.0</td>
<td>11.063</td>
<td>11.067</td>
<td>0.00</td>
</tr>
</tbody>
</table>

### Figure 1

Figure 1. Evolution of the solution with respect to the maximum solution time.

### 3.8. Hybrid approach

Another approach based on metaheuristic techniques has been developed. A solution is represented by the matrix \( \delta_{ijj'} \), where \( \delta_{ijj'} = 1 \) means that the train \( i \) at platform \( j \) share energy with the train \( i' \) at platform \( j' \). The hybrid algorithm includes a linear programming (LP) phase where the timetable is obtained as the local optimum with these binary variables as known. At the beginning solution population is provided by the MIP problem since it is very difficult to obtain feasible solutions with a genetic algorithm. Other initial solutions are obtained by mutation. A mutation operator and crossover operator have been implemented. The mutation operator takes the best timetable obtained so far and changes some entries. The crossover operator is applied to two timetables (randomly selected from a population of ten timetables). Almost no solution is obtained with the crossover operator since it is very difficult to combine two given solutions and obtain a new feasible one. The parameters of the hybrid model are the number of mutations and the probability of mutation.

In the following table some results of this hybrid approach for different maximum solution times and for different initial population are presented.
It can be observed that the improvement of the MIP optimal solution is low with respect to the maximum solution time, especially in the neighbourhood of the optimal solution. The difficulty in solving a MIP problem is measured by the relative tolerance or integrality gap. With the hybrid approach improvements of the solution are local (i.e., obtained only with the mutation operator), see lower red stepwise function of Figure 2 as a case where the hybrid algorithm is far from the optimal solution. However, it is very difficult for the MIP problem to prove optimality, so a good combination of both approaches can be worthwhile and would be the use of the hybrid algorithm when the solution of the MIP problem does not improve enough for a long time determined by the user (see upper red stepwise function of Figure 2 as a case where the hybrid approach is used at the end of the MIP approach).

4. Case study

This train timetabling model has been tested with a realistic case corresponding to line 3 of Metro de Madrid. As mentioned in the introduction, the optimization problem may be used for:

1. Evaluation of the overlapping time for the initial arrival and departure times

   The initial timetable contains 2 hours of overlapping time (which represents 53 % of the total slow-down processes).

2. Maximizing overlapping time keeping the published commercial timetable

   In this case the overlapping time reached 2.9 hours (within a solution time of 100 s).

These results show that coincidence time can be increased dramatically by train synchronization and, therefore, energy saving. Although the MIP solution obtained in a
certain solution time cannot be proved to be optimal it represents an important improvement in the objective function with respect to the overlapping time of the initial timetable. Besides, the overlapping time can be achieved with no modifications of the current published timetables.

The following coloured table shows where the changes in the timetable have been made for the second case mentioned before. Because of the huge amount of data, only the platforms connected to a given substation are shown. Each pair of synchronized trains are labelled with the same colour. Non-synchronized processes are represented in white.

<table>
<thead>
<tr>
<th>Calculated Schedule</th>
<th>Platform</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LV1</td>
</tr>
</tbody>
</table>

Table 1. Coincident trains when the timetable is calculated optimizing only departure times.

A detail of one of the synchronizations achieved is represented in the next figure:

![Image of Figure 4](image-url)

Figure 4. Detail of one of the synchronizations achieved

In the following table, the time differences of initial and final timetables are presented. The intensity of the colour is related with the overlapping time. The main schedule changes are in the middle of the table and need to be anticipated several platforms in advance, what constitutes an important result that was not previously suspected.
Table 2. Time differences between timetables \( \Delta d_j = 0 \) because early departures are not allowed (the published service must be guaranteed for every passenger), only delays of the trains).

The cumulative distribution function of the overlapping time for the second case is depicted in figure 4. The high frequency of the maximum value (25 seconds) means that small incidences are nearly imperceptible for the proposed timetable. Many overlapping times correspond to synchronizations greater than 15 seconds.

5. Conclusions

The timetabling model presented in this paper is used for maximizing the overlapping time between the slow-down and speed-up processes of underground trains in order to achieve energy savings.

The model presented in this paper is a decision support tool that can be used for maximizing the overlapping time between the slow-down and speed-up processes of underground trains in order to achieve energy savings. This problem has been formulated as a MIP problem with a
combinatorial nature and difficult to solve. However, quasioptimal solutions can be obtained in a reasonable amount of time and show the dramatic potential savings achievable.

Little improvements of the solutions are obtained with the hybrid approach, since it is very difficult to obtain new feasible solutions combining old ones.

In these case study all the overlapping time is achieved without modifying the current published timetable and, therefore, with no public knowledge.

This model can be easily extended to consider several underground lines supplied by some coincident electrical substations or a wider time scope.

Acknowledgements

The authors of the paper acknowledge the collaboration of Metro de Madrid in providing data and following the model development.

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